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Applying Information-Gap Decision Theory to a Design Problem having Severe Uncertainty

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ABSTRACT

Often in the early stages of the engineering design process, a decision maker lacks the information needed to represent uncertainty in the input parameters of a performance model. In one particular form of severely deficient information, a nominal estimate is available for an input parameter, but the amount of discrepancy between that estimate and the parameter's true value, as well as the implications of that discrepancy on system performance, are not known. In this paper, the concepts and techniques of information-gap decision theory (IGDT), an established method for making decisions robust to severely deficient information, are examined more closely through application to a design problem with continuous design variables. The uncertain variables in the chosen example problem are parameters of a probability distribution, so the relationship between IGDT and design approaches considering precise and/or imprecise probabilities is explained. Insight gained from a walkthrough of the design example is used to suggest the types of problems an IGDT approach will or will not effectively solve as well as potential limitations that could be encountered when solving more complex problems.

INTRODUCTION

Increasingly, engineers designing complex systems are forced to make design decisions amidst sizable *epistemic* uncertainty, that due to lack of information. Inability to collect information can be due to the expense of testing experimental or computational performance models, shortened or concurrent product development timelines, imprecise design specifications, lack of knowledge about the environment in which the design will be deployed, and so forth. Epistemic uncertainty, which could be reduced if resources were available for collecting more information, often exists in combination with chance variation, or *aleatory* uncertainty, which is effectively irreduceable and can be represented probabilistically.

There are a variety of design approaches that factor the effects of epistemic uncertainty into decision making. These include the use of safety factors [1]; design methods utilizing intervals with bounds derived either subjectively [2] or from data samples; and safety analysis using Bayesian probability distributions based on a combination of experts' subjective intervals, assumed priors, and estimates of expert error [3]. More recent approaches have identified and addressed design situations with distinguishable epistemic and aleatory uncertainty components. One example is Dempster-Shafer evidence theory [3-5], which captures imprecise information about variation using belief and plausibility measures derivable from a variety of information sources. One very recent approach [6] has utilized probability bounds analysis [7], combining concepts from probability theory and interval analysis. All of the preceding methods rely on either multiple data samples or subjectively defined distributions or intervals based on expert knowledge.

There are circumstances, however, where there is nothing available to describe an uncertain variable other than a nominal estimate (e.g., a comparable baseline, expert estimate, etc.), with upper and lower error bounds on either side of that estimate unknown. Common responses to having only such severely deficient information include postponing decision making entirely, haphazardly collecting more data without a clear understanding of its value, or even relying on unwarranted assumptions that fill in missing information.

Information-gap decision theory (IGDT), developed by Ben-Haim [8], is an alternative approach to making design decisions when there is an unknown gap between an uncertain quantity's available (but suspect) nominal value and its true value, the latter of which *could* be known but is not. IGDT models the size of the gap between the known and unknown as a free *uncertainty parameter*, α. To confront this gap, the design decision maker must specify a *satisficing* performance level—a "good enough" minimum level acceptable in a worst case scenario—and accordingly choose the design that, subject to that survival requirement, safely allows for the greatest amount of error[1], i.e., the largest α. This choice is based on a satisficing, robustness-maximizing decision rule, which can be preferable to a performance-optimizing rule applied amidst deficient information. The fact that the uncertainty parameter α is initially unspecified, with robustness to its unknown size maximized in the search across the design space, makes IGDT different than other decision approaches.

IGDT has steadily evolved over 15 years from a body of work on convex set-based models of uncertainty [10-12] and has been used in a variety of applications, including flood management [13], water resources management [14], correlation studies between experimental tests and simulations [15], structural design [16,17], and biological conservation management [18]. However, thorough, illustrative examples of the method, its mechanics, and its implications are still somewhat limited, especially for design problems with continuous design variables.

In this paper, an application of IGDT to a pressure vessel thickness design problem is presented to explore the concept and implications of trading risky optimized performance for info-gap robustness. In the problem, material strength inherently has aleatory uncertainty representable by a probability distribution; however, the designer lacks experimental samples needed to find the mean and standard deviation parameters of that distribution, so there is also epistemic uncertainty. An info-gap analysis is conducted across the design space to explore the capacity for robustness to error estimating probabilistic parameters while still guaranteeing some level of expected profit. Rather than trying to add to info-gap theory, the purpose of this paper is to walk through a design example and suggest the types of problems an IGDT approach will or will not effectively solve as well as potential limitations that could be encountered when solving more complex problems. Because uncertainty in the parameters of a probabilistic distribution is modeled, the relationship between IGDT and design approaches that consider both precise and imprecise probabilities is explained.

IGDT CONCEPTS AND COMPONENTS

Instead of optimizing performance, IGDT optimizes a *robustness function* subject to a satisficing constraint on performance or reward[2]. Satisficing means accepting designs with "good enough" performance in order to afford the potential to attain other objectives, especially when only idealized models or limited information is available [19]. Using IGDT, one satisfices performance to increase immunity to error due to unavailable information about bounds on an uncertain variable. The robustness function, $\hat{\alpha}(q,r_c)$, quantifies the uncertainty level that can be sustained while still guaranteeing that a desired critical reward is met. By maximizing the robustness function over the decision space, one finds the "robust-optimal" design, which is the most robust to epistemic uncertainty of unknown size. The alternative to the robust-optimal choice is the performance-optimal choice normally sought in optimization.

All of the theory foundational to IGDT and presented subsequently in this section can be found in [8]. The three components needed for an info-gap analysis are:

1. A performance (or "reward") model, $R(q,u)$, of system response that is a function of an uncertain variable, u, and some design variable(s), q; and whose output is a performance attribute of interest.
2. u, the uncertain variable that can be modeled as an info-gap and relates to (1) above.
3. r_c, a critical satisficing value of performance that must be guaranteed; alternatively considered a failure criterion.

In IGDT, it is assumed that even the size of the epistemic uncertainty is unknown — one only knows that there is uncertainty associated with a particular quantity, and knows an estimate of the nominal value for that quantity, but does not know the size of the uncertainty for that quantity. As shown in Figure 1, uncertainty, u, is represented as nested, convex sets centered[3] around a nominal value, $\tilde{u}$. The size of each set is characterized by the free uncertainty parameter, α. Mathematically, a simple uniformly bounded info-gap can be defined as:

$$u = \mathcal{U}(\alpha,\tilde{u}) = \left\{ u : \left| u - \tilde{u} \right| \le \alpha \right\}, \alpha \ge 0 \qquad (1)$$

Info-gap models are defined based on information about how the bounds on the uncertain variable grow. Besides the uniform bound model of Eq (1) and Figure 1, info-gaps can be bounded using various envelope types, integrals, Fourier bounds, etc., as discussed in [8].

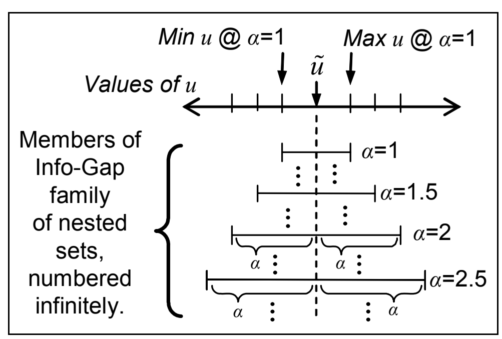

Figure 1: Representing unbounded uncertainty as an α–parameterized family of nested sets

[1] While Ben-Haim refers to this immunity to error as "robustness", some, e.g., [9], would favor the term "reliability". This paper stays consistent with Ben-Haim's terminology.

[2] The term "reward", favored by Ben-Haim, will be used interchangeably with the term "performance" in this paper.

[3] The info-gap model, parameterized from its center, has two ends of interest for each set in the family, as seen in Figure 1. The focus of this paper will only be on the bound that creates the worst consequence to performance. However, IGDT can consider the "better" end of the interval when using an *opportunity function* [8], not discussed herein.

From the three IGDT components, a robustness function can be defined that maximizes the size that the uncertainty parameter α can take as still satisfy the satisficing constraint. When increased $R(q,u)$ is desirable, the satisficing constraint is:

$$R(q,u) \geq r_c \tag{2}$$

This constraint is embedded into the robustness function, defined mathematically as an optimization problem:

$$\hat{\alpha}(q,r_c) = \max\left\{\alpha : \min_{u \in U(\alpha,\hat{u})} R(q,u) \geq r_c\right\} \tag{3}$$

Read aloud, info-gap robustness is "the maximum tolerable α so that all u [in the info-gap model's family of sets] up to uncertainty size α satisfy the minimum requirement for survival" [8]. The "hat" on the symbol for robustness, $\hat{\alpha}$, distinguishes it from uncertainty size α. The actual value of α is unknown, but one can still determine how much robustness, $\hat{\alpha}$, to deviation between the known nominal and unknown actual can be gotten by choosing a satisficing design rather than a risky performance-optimal one.

If the satisficing constraint, r_c, is flexible, one can examine the effect that relaxing the requirement has on opportunity for info-gap robustness. By graphically plotting, as in Figure 2, robustness to uncertainty versus demand for minimum satisficing performance, one can analyze the tradeoff. (Though not shown, every point along the curve has associated a unique robust-optimal design, $\hat{q}(r_c)$.) In Figure 2, it is apparent that relaxing one's requirement for minimum satisficing performance takes advantage of an accelerating payoff in robustness to info-gap uncertainty.

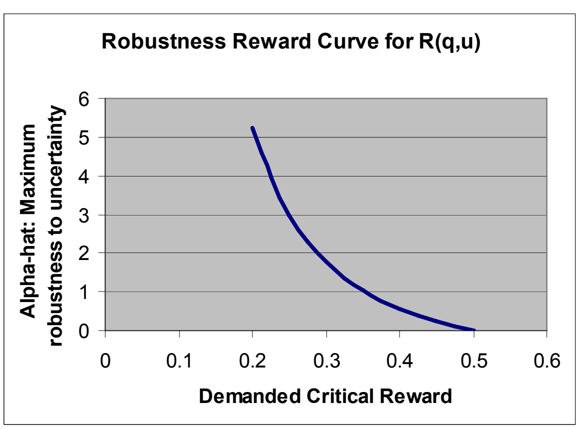

Figure 2: Typical tradeoff between achievable robustness and requirement for guaranteed satisficing performance

To review, the typical steps to finding a satisficing, robust-optimal design using IGDT include translating the severely uncertain information into an info-gap model, defining the reward function, $R(q,u)$, choosing a critical level of guaranteed performance, r_c, and finding the robust-optimal design, $\hat{q}(r_c)$. If the requirement for critical performance is flexible, one can take the additional step of plotting the relationship between r_c and $\hat{\alpha}(r_c)$. Beyond the preceding descriptions, it is the belief of the authors that the most effective way to introduce the IGDT approach is to apply it to a problem.

EXAMPLE APPLICATION WITH PROBABILISTIC UNCERTAINTY

A pressure vessel design problem featuring probabilistic uncertainty is used to demonstrate how to find preferable designs in scenarios where information describing the uncertainty is decreasingly available. The relatively simple problem has been adapted from the example in [6]. This section defines the design problem and solves it for a series of cases, first with no uncertainty, and then with probabilistic uncertainty using different starting information about distribution parameters.

DESIGN SCENARIO

A cylindrical, hemispherically ended pressure vessel is to be designed per the specifications shown in Figure 3. Geometric requirements constrain the problem so that the design can be specified with only one independent variable, the wall thickness, *t*, which is uniform throughout the vessel.

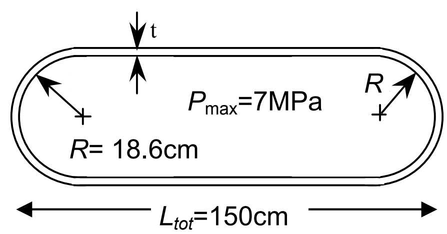

Figure 3: Longitudinal cross section of a pressure vessel, adapted from [6]

A performance function, to be optimized over the range of feasible thicknesses, is expressed in terms of profit:

$$Profit = Price - (Cost_{material} \cdot Vol_{material}(t)) - Cost_{failure}(t, S_y) \tag{4}$$

where:

t	= vessel wall thickness, design variable
S_y	= yield strength of pressure vessel
$Price$	= \$200
$Cost_{material}$	= material cost per volume = \$8500/m^3
$Vol_{material}(t)$	$= \pi\left(\frac{4}{3}\left(\left(R+t\right)^3 - R^3\right) + L_{cyl}\left(\left(R+t\right)^2 - R^2\right)\right)$
$Cost_{failure}(t, S_y)$	$= \begin{cases} 0 \text{ if } S_y \geq \sigma_{max}(t); \\ \$1{,}000{,}000 \text{ otherwise} \end{cases}$ (5)

Maximizing the profit function yields the optimal trade-off between the cost of the material and the cost of failure, with the latter set at US\$1,000,000 to cover property damage, personal injury, etc. If the strength is deterministic, e.g., S_y=180MPa, finding the optimal design entails finding the thickness where hoop stress in the vessel walls equals material strength:

$$S_y = \sigma_{\max} = (P_{\max} \cdot R)/t \rightarrow t = (P_{\max} \cdot R)/S_y = 0.72\text{cm}$$

This thickness, which barely avoids failure, allows a profit of \$90.65. It will be assumed henceforth, however, that material strength has variability due to randomness in the manufacturing process. A designer having sufficient experience with both the size of the variability and the implications of failure could assign a safety factor, e.g., $\tilde{S}_y \geq SF \cdot \sigma_{\max}(t)$, and accordingly multiply the optimal thickness from the deterministic case by the safety factor to find a safe design that would not incur the failure penalty. It will be assumed that the designer lacks the experiential information and design insight required to choose such a safety factor satisfactorily.

DESIGN SOLUTIONS USING PROBABILISTIC REPRESENTATIONS

Finding a reliable design when strength is variable is slightly more complex. Throughout the study at hand, true uncertainty in material strength takes the form of a normal (Gaussian) distribution, $S_y \sim Normal(\mu_S, (\sigma_S)^2)$, having parameters of mean, μ_S, and standard deviation, σ_S, as depicted in Figure 4. A reliable design will have a thickness large enough to assure that material stress is lower in magnitude than the bulk of probable strength occurrences, i.e., to the left side of the distribution in the figure. An *optimal* design taking risk (due to variability) into account can be found by using the cumulative density function in calculations for maximizing the *expected value* of the profit function of Eq. (4) (which implies risk-neutral preferences), thereby balancing expected failure penalty with excess material cost. The expected value operator, $E(X)$, also referred to as the expectation, is the statistical operator that weights the magnitude of each possible outcome with its probability of occurring [20]. For notation convenience henceforth, to avoid excessive use of parentheses, the prefix "E" will designated the expected value, e.g., *EProfit*.

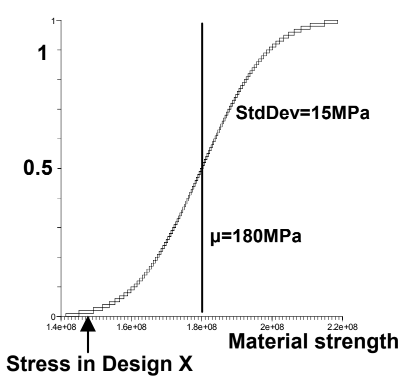

Figure 4: Perfect representation of strength variation as a cumulative distribution function.

Solutions using a distribution with precise parameters

Solving for the optimal design is relatively simple if the true uncertainty is known. Given the precise distribution $S_y \sim Normal(180MPa, (15MPa)^2)$, the optimal thickness is found to be 1.14 cm, yielding an expected profit, *EProfit*, of \$20.52. However, defining a perfectly precise probabilistic representation of uncertainty requires either infinite test trials or a subjective estimate based on expert knowledge. Whether or not a representation can be considered "precise enough" depends on the severity of consequences created by discrepancy between the estimated and true probability distribution tails.

Solutions using a distribution with imprecise parameters

In the study by Aughenbaugh and Paredis [6], the pressure vessel design problem was solved both with and without the assumption of a precise probabilistic representation. Values for S_y were known to be normally distributed but only a finite number of independent (random) tension tests were available. In one representation, a precise distribution was fit to limited data. In another, the authors represented imprecision using a construct from probability bounds analysis: the probability-box, or "p-box" [7]. Shown in Figure 5, the p-box bounds an interval in which, consistent with the amount of test data available, the actual distribution could assume one of any number of curves. (The unknown actual is depicted in the center in Figure 5, between the p-box bounds.) When a "maxi-min" decision rule is applied, performance is optimized using the tails of the conservative (left) side of the p-box; a lower stress is achieved with a thicker design. In the study, it was found that when fewer than 75 test trials were available, using the p-box and maxi-min decision rule had the result of, *on average*, yielding designs with expected profit values greater than those of designs found using precise normal distributions. (Though it is desirable, for comparison, to know the thicknesses found using p-boxes, design solutions varied depending on what values data took when randomly sampled from a true normal distribution.)

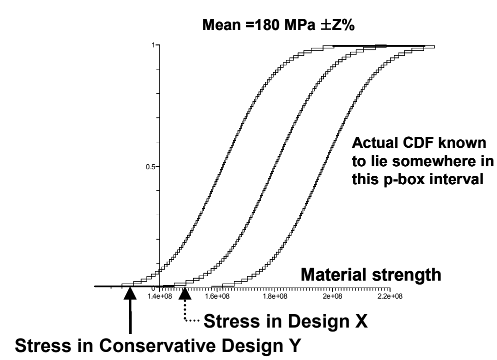

Figure 5: Probability distribution with imprecise parameters (p-box), and the resulting conservative design

Dilemma: deficient info about distribution parameters

What if information is so deficient that bounds on imprecision in uncertain parameters cannot be determined? The approaches reviewed thus far require either experiential information relating uncertainty to consequences, as with a safety factor, or numerous test samples, as when fitting a precise distribution or bounding the distribution imprecision with a p-box. When neither type of information is available, using those approaches demand that the decision maker rely on unwarranted assumptions about the true (or worst case) value of the mean, μ_S, and standard deviation, σ_S, of the probabilistic distribution. Design decisions made without some analysis of the implications of error in those assumptions have been found to lead to catastrophic failure [10].

SOLUTION WITH ROBUSTNESS TO INFO-GAPS

In the following sections, the info-gap approach will be used to design for maximum immunity to an information gap in the form of deviation between the unknown true variability and a "best-guess" estimate of what it might be. To avoid unwarranted assumptions about unavailable data, only following information will be utilized:

- That uncertainty in material strength is normally distributed with unknown mean and standard deviation parameters,
- That "best-guess" nominal estimates of those parameters can be taken from a similar material,
- An equation for system performance, *EProfit*, and
- How bounds on the uncertainty grow (which for this example is as a simple, uniformly-bounded interval), even though the extent of that growth is unknown.

Employing the info-gap approach to design entails evaluating what implications error in parameter estimates have on design performance. Given only severely deficient information as described, the designer decides that focusing on a *satisficing* level of performance, if guaranteed, is acceptable and preferable to risky optimized performance. In this walkthrough of an info-gap analysis, the benefits of choosing designs with satisficed rather than maximized performance are highlighted, and the relationship between demanded performance and info-gap robustness are revealed graphically. The results from all of these design activities are summarized at the end of this section.

INFO-GAP MODEL

How an information-gap model should be defined to best represent uncertainty depends on the information available. Though one could represent variability in strength wholly (i.e., not by its parameters) with an info-gap model, doing so ignores the knowledge that the uncertain variable is normally distributed and ignores the failure penalty defined in Eq. (5). In a different and better approach, Ben-Haim (2001, Chapter 11) has defined a "hybrid uncertainty" info-gap model for deviation in knowledge about the tails of a distribution. To the authors' knowledge, the hybrid info-gap technique has not, however, been shown applied to a problem, and it is questionable how accurately one could, in practice, judge the need for robustness to such deviation in tails when expressed in that way.

For the study at hand, an information-gap model of uncertainty in material strength distribution is defined per the following scenario. The designer has a new material available from which the pressure vessel will be made. Due to the expense of the material, or perhaps due to the fact that it is still being designed, there are no tensile-test trials available to characterize its material strength. However, experience with a similar baseline material suggests that, due to variation in its manufacturing, values for material strength have normally distributed variability. Additionally, adequate tensile-test data exist for the baseline material, and from that data reasonably precise baseline values for mean, $\tilde{\mu}_s$, and standard deviation, $\tilde{\sigma}_s$, normal distribution parameters are available. The info-gap model used in this design activity will utilize information from the baseline material, even though the equivalence of its material properties to that of the new material are unknown (or cannot be expressed accurately). The decision maker is certain that the new material's uncertainty distribution is normal, per $S_y \sim Normal(\mu_S, (\sigma_S)^2)$. However, there is unknown deviation between the baseline material's known distribution parameters and the unknown distribution parameters of the new material. The unknown size of that discrepancy is the designer's information gap.

Information-gap models for μ_s and σ_s can be defined as follows. The existing material's mean and standard deviation parameters, $\tilde{\mu}_s$ =180MPa and $\tilde{\sigma}_s$ =15MPa, will serve as the nominal value(s) of the info-gap model for the new material. A decision maker has the modeling choice of whether to express deviation between the nominal values and actual values either as an absolute quantity or a "fractional" percentage of the nominal value. It is assumed that the latter form is more intuitive form for judging deviation. The info-gap models are each a family of nested sets, with each family member's size corresponding to a value of the uncertainty parameter, α. Expressed mathematically:

$$\mu_S(\alpha,\tilde{\mu}_S) = \left\{ \mu_S : \left| \frac{\mu_S - \tilde{\mu}_S}{\tilde{\mu}_S} \right| \le \alpha \right\}, \alpha \ge 0 \qquad (6)$$

$$\sigma_S(\alpha,\tilde{\sigma}_S) = \left\{ \sigma_S : \left| \frac{\sigma_S - \tilde{\sigma}_S}{\tilde{\sigma}_S} \right| \le \alpha \right\}, \alpha \ge 0 \qquad (7)$$

The units for α are in percent of the nominal value. Restrictions can be placed on the family such that negative μ_s or σ_s are avoided. In simpler terms, each info-gap model can also be defined as, e.g.,:

$$\tilde{\sigma}_S(1-\alpha) \le \sigma_S \le \tilde{\sigma}_S(1+\alpha) \quad (8)$$

To provide some perspective, parameterizing the imprecision bounds on μ_s with an info-gap makes the model for the distribution look like a p-box with outward expanding bounds, as depicted in Figure 6.

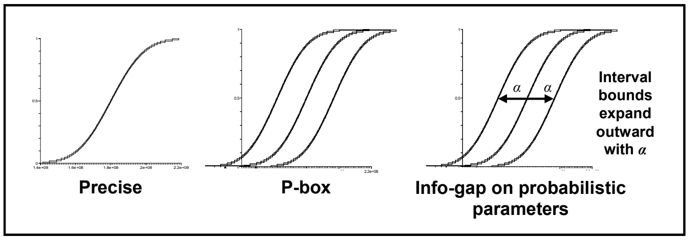

Figure 6: Modeling imprecision in distribution parameters

In the following section, for the sake of simplicity, the IGDT approach will be implemented for one uncertain variable, σ_s, the standard deviation of the distribution. Later in the Evaluating Multiple Uncertainties section of this paper, the problem will be analyzed for both unknown variables simultaneously.

REMAINING INFO-GAP PROBLEM FORMULATION COMPONENTS

The remaining components of the info-gap problem definition can now be defined. The reward function can be found by taking the expected value of the function of Eq. (4), and expressing it in the form of Eq. (2), with the decision vector q set as the design variable t and the uncertain variable u set as the uncertain standard deviation σ_s, of which material strength S_y is a function:

$$\begin{aligned} R(q, u) &= EProfit(t, S_y(\sigma_s)) \\ &= \mathrm{E}[Price - (Cost_{material} \cdot Volume_{material}(t)) \\ &\qquad - Cost_{failure}(t, S_y(\sigma_s))] \end{aligned} \quad (9)$$

Project managers take the satisficing attitude that the pressure vessel will not be viable economically if its expected profit is less than $EProfit_{critical}$=US$14.10. Per the form in Eq. (2), this critical satisficing constraint is:

$$EProfit(t, S_y(\sigma_S)) \ge EProfit_{critical} \quad (10)$$

The info-gap robustness, $\hat{\alpha}(t, EProfit_{critical})$, is, for a thickness *t*, the maximum size (in percent deviation from nominal) that the uncertainty parameter α can grow and still guarantee *at least* the chosen satisficing expected profit, $EProfit_{critical}$, the minimal requirement for survival. Expressed mathematically in the form of Eq. (3):

$$\hat{\alpha}(t, EProfit_{critical}) = \max\left\{\alpha : \min_{\sigma_S \in U(\alpha, \tilde{\sigma}_S)} EProfit(t, S_y(\sigma_S)) \ge EProfit_{critical}\right\} \quad (11)$$

Notice that the constraint from Eq. (10) is embedded in this optimization problem. Out of the possible range of the design variables, some thicknesses will be too small to satisfy the constraint at all (as $\hat{\alpha} < 0$ is defined as invalid); one thickness value will satisfy the constraint but have no robustness to info-gap uncertainty at all ($\hat{\alpha} = 0$); some will offer robustness ($\hat{\alpha} > 0$). Of the latter group, one thickness will have the largest $\hat{\alpha}$, making it the robust-optimal design solution, $\hat{t}(EProfit_{critical})$, a value constrained by the choice of demanded critical reward.

When $EProfit_{critical}$ is a free parameter, a robustness-performance tradeoff curve is created, as will be shown later. In the next section implications of satisficing at the fixed level $EProfit_{critical}$=US$14.10 will be graphically explored.

SOLVING FOR THE ROBUST-OPTIMAL DESIGN

Because it is cumbersome to combine Eqs. (7) and (9) and symbolically solve for an info-gap robustness equation in the form in Eq. (11), simple exhaustive computation of expected profit across a range of $\alpha > 0$ has been used to find numerically the functional relationship between uncertainty and reward. While most IGDT examples in the literature find and plot $\hat{\alpha}(r_c)$ rather than $R(\alpha)$ as in our exhaustive technique, the latter technique is valid and produces identical plots that can be used to find $\hat{\alpha}$ (robustness to maximum uncertainty) at a given demand for $EProfit_{critical}$.

The uncertain σ_s in Eq. (9) is replaced with the info-gap model form of Eq. (8), concentrating on the side associated with increasing failure risk, $\sigma_S \le \tilde{\sigma}_S(1+\alpha)$, to make expected profit become:

$$\begin{aligned} &EProfit(t, S_y(\sigma_s)) \\ &\quad = \mathrm{E}[Price - (Cost_{material} \cdot Volume_{material}(t)) \\ &\qquad - Cost_{failure}(t, S_y(\tilde{\sigma}_S(1+\alpha)))] \end{aligned} \quad (12)$$

A plot of expected profit versus thickness is shown in Figure 7 for different values of α. When α=0, the condition where there would exist no potential deviation between σ_s of the new and baseline materials, the maximum expected profit of US$20.50 is obtained at a design thickness of 1.14cm. Stated in the normal IGDT fashion: a designer demanding a non-satisficing level of $EProfit_{critical}$≥US$20.50 cannot achieve robustness to any growth in α; in other words, $\hat{\alpha}(\$20.50)$=0. As the demand for guaranteed performance relaxes, i.e., is satisficed, uncertainty can grow larger (as shown by the progression of curves with lower peaks moving to the right of Figure 7) and still have a set of designs that will guarantee that performance or better. The value of α that shifts and lowers the peak of the curve until it exactly meets the satisficing demand is the robust optimal solution. For instance, if the designer demands

$EProfit_{critical}$≥US$14.10, the error in the estimate of σ_s can grow as great as α=6%. In other words, $\hat{\alpha}(\$14.10)$=6%, attainable with a robust-optimal thickness of $\hat{t}(\$14.10)$=1.18cm. This is indicated on the innermost curve in Figure 7. Any smaller error (e.g., as shown by the profit curves generated at 2% and 4% estimation error) encountered by $\hat{t}(\$14.10)$ can only perform better than the demanded minimum.

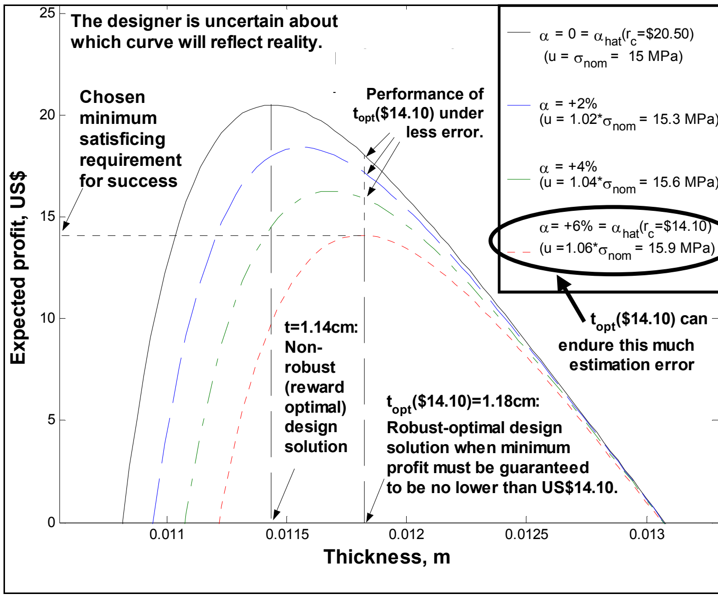

Figure 7: *EProfit* (performance) vs. thickness (design variable) for various *α*

The strategy of choosing the design variable that maximizes robustness to info-gap uncertainty has positives and negatives, depending on the true realization of deviation between $\tilde{\sigma}_S$ and σ_S. As can be seen in Figure 8, the thickness that is optimal under no uncertainty, α=0, will perform worse than the robust-optimal solution $\hat{t}_{\hat{\alpha}=0.06}$, by a difference in expected profit of $4.33, if the error in estimating standard deviation turns out to be 6%. That loss is due to the bold behavior of requiring no robustness to deviation between the estimate and reality. Conversely, should the error turn out to be α=0, the robust-optimal design achieves US$2.40 less than the achievable maximum expected profit. Thus, there is an "insurance cost" of robustness should the info-gap be smaller than expected.

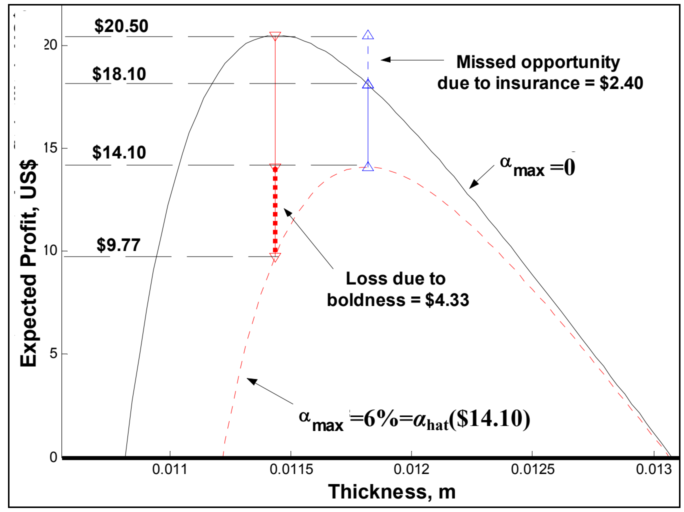

Figure 8: The benefits and sacrifices of satisficing to gain immunity to uncertainty of unknown size.

ELICITING PREFERENCES FOR THE ROBUSTNESS-REWARD TRADEOFF

Whereas the aim in the previous section was to find the design robust to an info-gap as much as the satisficing requirement $EProfit_{critical}$ would allow, a decision maker can also analyze how much robustness can be gained by relaxing that critical requirement. This involves plotting the robustness function $\hat{\alpha}(t, \mathrm{E}Profit_{critical})$ of Eq. (11) across a range of $EProfit_{critical}$ values, and then, from that plot, eliciting a preference for the robustness-performance tradeoff.

The curves shown in Figure 7 and Figure 8 are actually slices from a three-dimensional relationship between thickness, *t*, the uncertainty parameter, α, and the expected profit, satisficing tolerance for which is chosen by the decision maker. These relationships are shown in Figure 9, with contours (the third dimension) of expected profit.

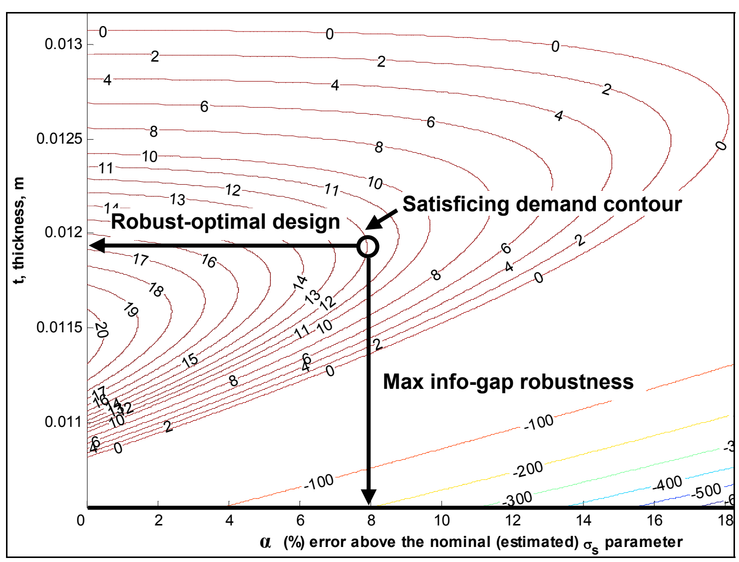

Figure 9: Contour plot of *EProfit* vs. uncertainty , *α*, and *t*

The condition of $\hat{\alpha} = 0$, where no robustness is demanded, can be seen again along the left border of Figure 9, where expected profits as high as US$20.50 are attainable, at a thickness of 1.14cm. Only a small range of designs can achieve an expected profit above US$20, and the most robust of those can only endure ~0.5% error in estimating the standard deviation parameter. Thus, the design space is not very robust to such estimation error.

The contours in Figure 9 are useful in locating the robust optimal design and maximum possible info-gap robustness at a given critical reward. The circle in the figure indicates the point where, along the contour of $EProfit_{critical}$≥US$12, the greatest info-gap robustness, to roughly 8% relative error in the estimate of σ_s, can be achieved. The robust optimal design is indicated to be at $\hat{t}$ =1.19 cm.

Whereas all relationships between t, E*Profit*, and $\hat{\alpha}$ can be seen in Figure 9, it is useful to focus on simpler plots of the robust-optimal relationships, as seen in the next two figures. The plot in Figure 10 is of the format typically employed in info-gap decision analyses to elicit preferences for tradeoff between satisficing requirement and robustness. Curvature in this plot, as compared to Figure 2, is barely perceptible, so the tradeoff is nearly linear and does not offer a significantly increasing rate of return with increased risk-taking. The optimal robustness that each thickness can obtain can be seen in Figure 11.

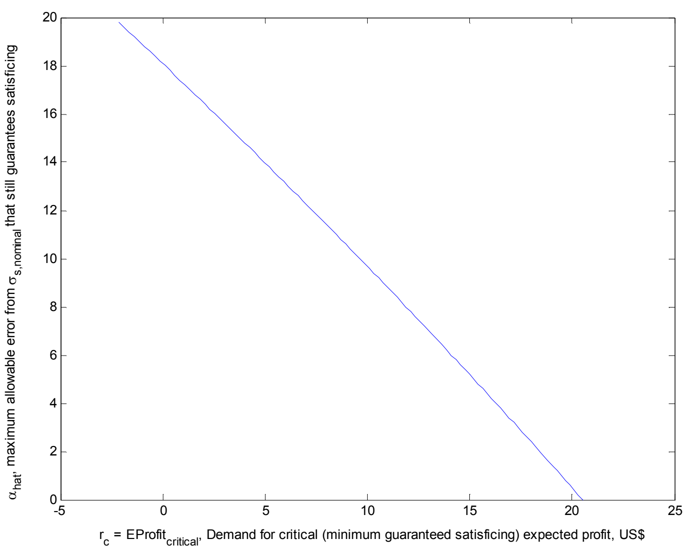

Figure 10: The standard robustness-reward plot

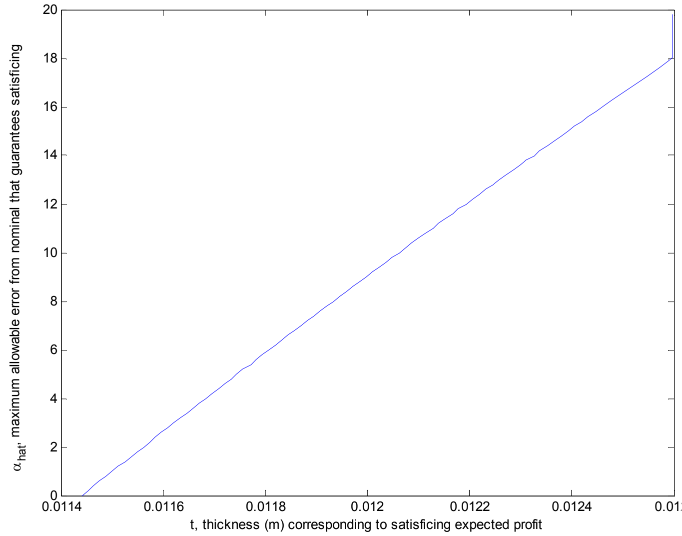

Figure 11: Maximum robustness attainable by different thicknesses

To summarize this section, the ability to analyze tradeoff preferences introspectively with the aid of the presented plots should increase a designer's comfort level in choosing a satisficing requirement by understanding the corresponding effects on design choice and the amount of robustness attainable, $\hat{\alpha}$.

EVALUATING MULTIPLE UNCERTAINTIES

One can simultaneously consider the effects of multiple uncertainties on design choices. Existing info-gap methods [8] entail defining the horizons of uncertainty for multiple uncertain parameters in terms of a single uncertainty parameter α. This technique is implemented for the modeling of info-gap uncertainty in both μ_s and σ_s as defined by Eqs. (6) and (7). The corresponding reward and robustness functions are defined as:

$$R(q, u) = EProfit(t, S_y(\mu_s, \sigma_s)) = \mathrm{E}[Price - (Cost_{material} \cdot Volume_{material}(t)) - Cost_{failure}(t, S_y(\mu_s, \sigma_s))] \tag{13}$$

$$\hat{\alpha}(t, EProfit_{critical}) = \max\left\{\alpha : \min_{\substack{\mu_S \in U(\alpha,\tilde{\mu}_S) \\ \sigma_S \in U(\alpha,\tilde{\sigma}_S)}} EProfit_{total}(t, S_y(\mu_S, \sigma_S)) \geq EProfit_{critical}\right\} \tag{14}$$

A plot of *EProfit*, shown in Figure 12, reveals the extent to which thickness must increase to account for the uncertainty added by error in estimating the mean strength relative to a nominal guess. Evaluating the effects of both uncertain parameters deviating from their nominal values at the same percentages may not be the most intuitive, but is the easiest to visualize on a simple contour plot. Comparing the magnitude of the curves to those in Figure 9, it can be seen that a percent shift in the mean parameter has significant negative impact on design feasibility. Much less robustness is attainable with a lack of information about the mean, even when thickness is increased to compensate for uncertainty.

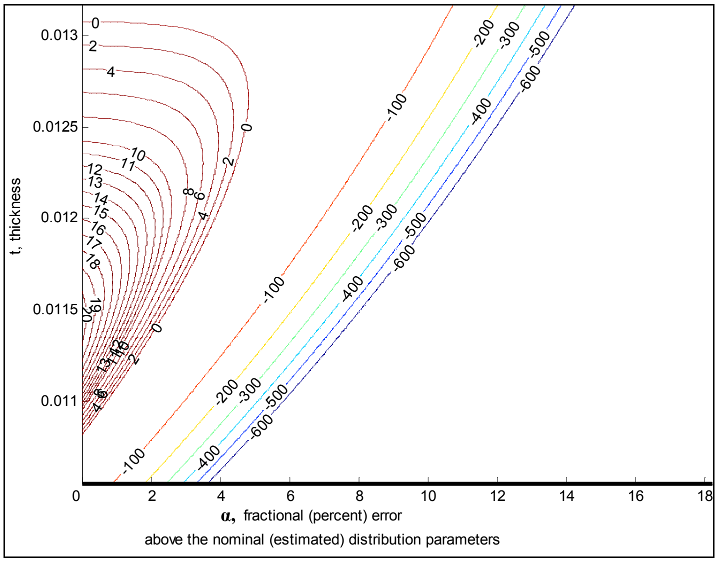

Figure 12: Contour of minimally attainable expected profit for *two* uncertain variables.

SUMMARY OF ROBUST-OPTIMAL DESIGN RESULTS

The following points review the engineering insight revealed in the preceding walkthrough of the info-gap robustness maximizing approach for probabilistic variables.

- By choosing to accept a design solution that guarantees no worse than $EProfit_{critical}$=US$14.10, instead of a profit maximizing solution based on nominal guesses whose accuracy is unknown, the designer can find a robust-optimal thickness that guarantees $EProfit_{critical}$ amidst as much as 6% discrepancy above the nominal estimate of standard deviation taken from a baseline material. Relaxing the demand for satisficing $EProfit_{critical}$ affords the opportunity for greater info-gap robustness if thickness is increased accordingly, as can be seen in Figure 7.
- In the case where the discrepancy between actual and nominal parameters turns out to be *smaller* than expected, an info-gap robust design performs less than a performance-optimal design, which has no robustness. In the pressure vessel example, there is US$2.40 in missed opportunity due to the "insurance cost" of robustness to 6% error above the estimated σ_S. However, if that discrepancy were to exist between the baseline and new materials, the loss incurred by a performance-optimal design, with respect to a robust-optimal design, is US$4.33, or over 80% more loss, as was illustrated in Figure 8.
- The tradeoff between robustness and performance occurs at a rate of a robustness increase of 0.9% (in added allowable deviation-from-nominal) for every expected US$1 sacrificed, as seen in Figure 10. Choosing a point on this tradeoff effectively requires a gamble when the size of uncertainty is unknown and a decision must be made. It should be noted, however, that being roughly linear, this tradeoff provides no greater or lesser rate of return at different levels of risk acceptance.
- According to Figure 10, for the given design concept parameterized by *t*, the designer cannot design in robustness greater than 18% error and expect to make a profit. If greater error is ever estimated to be a threat, no design in the given space will suffice; a new design concept is needed.
- Achieving robustness to info-gap uncertainty in the second uncertain mean parameter, μ_s, at deviation percentages equal to those for σ_s, is even less feasible. The design is more sensitive to error in estimating μ_s (as a percentage of the nominal guess), as can be seen in the relatively compressed scale of Figure 12.

DISCUSSION

In the following section, insight is provided about the conditions under which maximizing robustness to info-gaps is most advantageous, what pluses and minuses of the approach will likely be for other types of problems, and what future work can be identified.

WHEN TO USE THE IGDT APPROACH

The IGDT approach is an effective way to evaluate the potential for robustness to a starting guess of unknown error bounds. Using the techniques presented in this paper, one can identify a robust-optimal design from a range of continuous variables and evaluate the sensitivity of that design to increased demands on either info-gap robustness or minimally acceptable performance. In the pressure vessel example, the range of feasible thicknesses was very small, and none afforded a very large amount of robustness to deviation from the baseline. Thus, none of the designs would likely accommodate the robustness needed to guard against failure due to inaccurate nominal values.

In certain situations, the info-gap design analysis approach can eliminate the need for more complex techniques for design amidst uncertainty. An IGDT analysis can reveal that, for a range of design variables, achievable performance and corresponding robustness do not change considerably, as in cases when designs are either *always* or *never* able to succeed amidst severe uncertainty. In such cases, even when the chance to collect further information is an option, an IGDT approach is all that is necessary. This was found to be the case in the pressure vessel design problem, as only a narrow range of thicknesses yielded positive *EProfit*, and relaxing the requirement for success afforded very little robustness. Unless the designer foresees that parameter estimate error would never be greater than 18% above nominal, there is no guarantee that the design concept could expect to make a profit. The same conclusion would also have been reached using a more resource-intensive, sampling-based uncertainty analysis approach, but the info-gap analysis was implemented with only a search through a range of design variables and uncertainty extremes. Future work will attempt to more accurately quantify these cost savings.

In cases where the results are not so determinate for the entire range of a design variable, IGDT offers a path to a decision. If the range of feasible solutions yields both acceptable ranges of satisficing performance and reasonable levels of attainable robustness to info-gap uncertainty, one can choose a tradeoff between robustness and reward per the standard info-gap decision making approach. The act of choosing a point on the robustness-reward tradeoff is effectively a *gamble* about how much robustness is necessary. Outside of determining the optimal design normatively based on questionable assumptions and preferences established *a prioi*, this gamble is really the only course available when a decision *must* be made knowing only a nominal guess.

Maximizing info-gap robustness is not, however, the preferable approach if there is any opportunity to collect more information beyond just a nominal value. With the accumulation of actual data points, e.g. tensile tests on the new material, bounds on the maximum size of the discrepancy between nominal and actual uncertain parameters can be specified. In such a case, the info-gap uncertainty parameter, α, becomes fixed, and the info-gap model of uncertainty degenerates into a static interval. There is no need to favor a design with info-gap robustness nor evaluate any tradeoff between that robustness and satisficed performance. Instead, in the case where parameters μ or σ of a probability distribution are quantities with bounded imprecision, a p-box model of uncertainty is appropriate. The p-box should be used in conjunction with a compatible decision rule like "maxi-min", maximizing $EProfit$ on the minimum bound of the p-box interval. That "best worst case" rule was illustrated earlier in Figure 5, and was the strategy used in the example by Aughenbaugh and Paredis [6]. It should be noted that the contour plot from Figure 9, although generated to show the relationship between info-gap robustness and satisficing performance, can be also interpreted as maximum attainable performance for different p-box sizes, which correspond to positions on the horizontal axis.

LIMITATIONS AND FUTURE WORK

Though the info-gap robustness maximizing approach to design has promise in selected applications, there are limitations that hinder its adoption and inspire future work.

Intuitiveness of evaluating severe uncertainty and satisficing reward

The IGDT approach requires that the decision maker be able to set a critical satisficing performance target and, if needed, weigh and adjust that target in light of the potential for increased info-gap robustness. There may be scenarios where decision-makers are less able to evaluate and express such preferences. For instance, it is presumably more difficult to determine appropriate satisficing levels for performance aspects not expressed monetarily, e.g., environmental performance. Similarly, if a decision maker finds it difficult to relate to the magnitude of quantities like standard deviation in material strength, it could be difficult to judge the necessary level of info-gap robustness to error in that quantity.

The response to these concerns is that one must weigh their judgments as *preference for robustness size* (instead of estimation of uncertainty size, which at first seems only subtly different) in balance with preference for satisficing reward via a tradeoff curve such as in Figure 10. A discussion of preference calibration is examined in [8]; however, experimentation is needed to determine the success of decision makers in utilizing these techniques.

Difficulty of IGDT analysis in complex design scenarios

Analyzing the relationships between satisficing reward, info-gap robustness, and the robust-optimal design increases in difficulty whenever any of those components are a vector. In the example problem, analysis was aided by the plots in Figure 9 and Figure 10. However, having multiple variables for any of the components makes visualization and understanding of robustness-reward-design variable relationships and tradeoffs less intuitive. For instance, in the pressure vessel example, when simultaneously analyzing the influence of the two uncertain variables, μ_s and σ_s, a simplifying assumption was made that α, the percentage growth in deviation from their respective nominals, grew at the same rate for both variables. While using this assumption in the analysis revealed μ_s to have the greater influence, the coupling of the two uncertain variables through α was imposed for the sake of simplifying visualization and would not necessarily exist in reality. There is a need to intuitively relate and compare, for different uncertain variables, the respective expansion rates of the uncertainty bounds from nominal. It is expected that such information is not often available.

Furthermore, sensitivity testing of any other part of the problem scenario, e.g., testing the effects of changing the failure penalty in Eq. (5), would complicate analysis further. The success with which a decision maker could elicit preferences and choose designs amidst such complexity has not been evaluated, and very little software is available to facilitate the use of IGDT in analysis activities.

Computational issues

Lastly, computation issues were not encountered in finding robust-optimal solutions or plots but could become significant amidst numerous variables and/or complex design spaces with multiple local minima. IGDT computation issues in general have received limited attention, e.g., [21].

Comparison to other uncertainty representations

Experiments comparing competing approaches for design amidst epistemic uncertainty have been conducted for probability bounds analysis [6] and evidence theory versus Bayesian theory [3]. Although info-gap models are meant for use when much less information is available than is required by other uncertainty representations, it seems possible that there are still "gray areas" with regards starting information where it difficult to know which approach will produce the best results. Thus, future work will include experiments comparing IGDT results to that of other approaches. Additionally, it will be useful to consider what it might mean to transition between different design-for-uncertainty approaches as information is increased.

SUMMARY

Uncertainty encountered in the engineering design process can sometimes be so severe that only a nominal estimate exists for a parameter which, given much more information, could be known or at least placed within error bounds. In this paper, the details and concepts of the info-gap decision theory approach, which facilitates analysis of how to make decisions robust to deficient information, have been examined more closely through application to a problem with continuous design variables. Stated simply, the approach has been shown to help a designer increase their understanding of the effects of unknown imprecision in nominal estimates. Info-gap is relatively inexpensive to implement and more than adequate for some decision scenarios. A clear demarcation of the effectiveness of info-gap in practical situations, as well as closer examination of the method with respect to other robustness approaches, is left to future work.

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